DOI: 10.2478/auom-2020-0030
An. Şt. Univ. Ovidius Constanţa **Vol. 28(2),2020, 239–257**

On S-2-absorbing submodules and vn-regular modules

Gülşen Ulucak, Ünsal Tekir and Suat Koç

Abstract

Let R be a commutative ring and M an R-module. In this article, we introduce the concept of S-2-absorbing submodule. Suppose that $S \subseteq R$ is a multiplicatively closed subset of R. A submodule P of M with $(P :_R M) \cap S = \emptyset$ is said to be an S-2-absorbing submodule if there exists an element $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$, then $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. Many examples, characterizations and properties of S-2-absorbing submodules are given. Moreover, we use them to characterize von Neumann regular modules in the sense [9].

1 Introduction

In this article, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. The notion of prime submodule has an important place in commutative algebra and it is frequently used to classify the modules. Also, there have been many generalizations and study of prime submodules. See, for example, [5], [7], [10], [14], [16] and [17]. Recently, the authors in [18], introduced the notion of S-prime submodules which is a generalation of

Key Words: 2-absorbing submodule, S-2-absorbing submodule, von Neumann regular module.
2010 Mathematics Subject Classification: 13C05, 16E50, 16D60, 15A15.
Received: 31.08.2019
Revised: 25.09.2019
Accepted: 17.12.2019

prime submodules and they used the S-prime submodules to characterize certain class of modules such as torsion free modules and simple modules. Let $S \subseteq R$ be a multiplicatively closed subset, that is, S satisfies the following conditions: (i) $1 \in S$ and (ii) $s_1 s_2 \in S$ for each $s_1, s_2 \in S$. Recall from [18] that a submodule P of M with $(P :_R M) \cap S = \emptyset$ is said to be an *S-prime submodule* if there exists a fixed $s \in S$ such that $am \in P$ for some $a \in R$ and $m \in M$ implies that $sa \in (P :_R M)$ or $sm \in P$. In particular, an ideal I of R is an *S-prime ideal* if I is an S-prime submodule of R-module R. One of the important generalizations of prime submodule is the concept of 2-absorbing submodule. Recall from [5] that a proper submodule N of M is said to be a *2-absorbing submodule* if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Predictably, a proper ideal I of R is a *2-absorbing ideal* if and only if I is a 2-absorbing submodule of R-module R. The aim of this paper is to study the concept of S-2-absorbing submodule and use them to characterize von Neumann regular modules in the sense [9]. For the sake of completeness, will give some notations which will be used throughout this article.

Let N be a submodule of M, K be a nonempty subset of M and I be an ideal of R. Then we define the residuals of N by K and I as follows:

$$(N :_R K) = \{r \in R : rK \subseteq N\}$$
$$(N :_M I) = \{m \in M : Im \subseteq N\}.$$

Particularly, $(0 :_R M)$ is denoted by $ann(M)$. Also we use $(N :_M s)$ to denote $(N :_M Rs)$ for each $s \in R$ and we use $(N :_R m)$ to denote $(N :_R Rm)$ for each $m \in M$.

In this study, we introduce the concept of S-2-absorbing submodules of a module which is a generalization of S-prime submodules and 2-absorbing submodules. Also, this concept can be considered as a unification of S-prime and 2-absorbing submodules. A submodule P of M is said to be an *S-2-absorbing submodule* if $(P :_R M) \cap S = \emptyset$ and there exists a fixed $s \in S$ such that $abm \in P$ for some $a, b \in R$ and $m \in M$ implies that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. In particular, an ideal I of R is an *S-2-absorbing ideal* if I is an S-2-absorbing submodule of R-module R. Note that a 2-absorbing submodule P of M with $(P :_R M) \cap S = \emptyset$ is also S-2-absorbing but the converse is not true (See Example 1 and Example 3). Also note that if $S \subseteq u(R)$, where $u(R)$ is the set of units in R, then S-2-absorbing submodules and 2-absorbing submodules coincide (See Example 2). Among other results in this paper, in Section 2, we study the properties of S-2-absorbing submodules similar to 2-absorbing submodules. Also, we give the relations between S-prime and S-2-absorbing submodules (See Proposition 1 and Proposition 2). Furthermore, we investigate the behavior of S-2-absorbing submodules under localization, homomorphism, in trivial extension and in cartesian product of modules (See Proposition 1, Proposition 4, Proposition 6 and Theorem 3). In Theorem 1, we show that P is an S-2-absorbing submodule of M if and only if there exists an $s \in S$ such that $IJN \subseteq P$ for some ideals I, J of R and some submodule N of M implies $sIN \subseteq P$ or $sJN \subseteq P$ or $sIJ \subseteq (P :_R M)$. Recall from [8] that an R-module M is said to be a *multiplication module* if for each submodule N of M, there exists an ideal I of R such that $N = IM$, or equivalently, $N = (N :_R M)M$. In Theorem 2, we give a characterization of S-2-absorbing submodules in multiplication modules.

Section 3 is devoted to the study of von Neumann regular modules in the sense [9]. Recall from [21] that a ring R is said to be a *von Neumann regular (for short, vn-regular) ring* if for each $a \in R$ there exists an $x \in R$ such that $a = a^2x$. In this case, the principal ideal $(a) = (e)$ is generated by an idempotent element $e \in R$. Note that a ring R is a vn-regular ring if and only if $(a) = (a^2)$ for each $a \in R$. So far, the concept of vn-regular rings has been studied in many papers and has various generalizations. See, for example, [12], [13] and [22]. Recently, Jayaram and Tekir extended the notion of vn-regular ring to modules as follows: an R-module M is said to be a *vn-regular module* if for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$. In section 3, we first prove the Chinese Remainder Theorem for modules (See Theorem 5). Finally, we characterize vn-regular modules in terms of S-2-absorbing submodules (See, Proposition 9 and Theorem 6).

2 S-2-Absorbing submodules

Definition 1. *Let M be an R-module and S a multiplicatively closed subset of R. A submodule P of M is said to be an S-2-absorbing if $(P :_R M) \cap S = \emptyset$ and there exists a fixed $s \in S$ such that $abm \in P$ for some $a, b \in R$ and $m \in M$ implies that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$.*

Example 1. *Let M be an R-module and S a multiplicatively closed subset of R. Every 2-absorbing submodule P of M with $(P :_R M) \cap S = \emptyset$ is also an S-2-absorbing.*

Example 2. *Let M be an R-module and $S \subseteq R$ a multiplicatively closed subset consisting of units in R. Then a submodule P of M is 2-absorbing submodule if and only if P is S-2-absorbing.*

Example 3. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}_{pq}$, where $p \neq q$ are prime numbers. Then the zero submodule P is not 2-absorbing since $pq(0, \overline{1}) = (0, \overline{0}) \in P$ but $pq \notin (P :_{\mathbb{Z}} M) = 0$, $p(0, \overline{1}) = (0, \overline{p}) \notin P$ and $q(0, \overline{1}) = (0, \overline{q}) \notin P$. On the other hand, set $S = reg(\mathbb{Z})$ and put $s = pq \in S$. If $ab(x, \overline{m}) \in P$ for* *some $a, b, x, m \in \mathbb{Z}$, then $abx = 0$. Without loss of generality, assume that $a = 0$. Then $sa(x, \overline{m}) = (0, \overline{0}) \in P$. Thus P is an S-2-absorbing submodule of M.*

Let R be a ring and $S \subseteq R$ a multiplicatively closed subset of R. The saturation $S^\star$ of S is defined as $S^\star = \{x \in R : \frac{x}{1}$ is a unit of $S^{-1}R\}$. Note that $S^\star$ is a multiplicatively closed subset containing S.

Proposition 1. *Let M be an R-module and S a multiplicatively closed subset of R. Then the following statements hold:*

(i) Every S-prime submodule is an S-2-absorbing submodule.

(ii) Suppose that $S_1 \subseteq S_2$ are multiplicatively closed subsets of R. If P is an S_1-2-absorbing submodule and $(P :_R M) \cap S_2 = \emptyset$, then P is an S_2-2-absorbing submodule.

(iii) A submodule P of M is an S-2-absorbing submodule if and only if it is an $S^\star$-2-absorbing submodule.

(iv) If P is an S-2-absorbing submodule of M, then $S^{-1}P$ is a 2-absorbing submodule of $S^{-1}M$.

Proof. (i), (ii): It is clear.

(iii): Let P be an S-2-absorbing submodule of M. Assume that $(P :_R M) \cap S^\star \neq \emptyset$. Then we have $r \in (P :_R M) \cap S^*$. Then $\frac{r}{1}$ is a unit of $S^{-1}R$, that is, $\frac{r}{1}\frac{a}{s} = 1$ for some $a \in R$ and $s \in S$ since $r \in S^\star$. Hence $us = ura \in S$ for some $u \in S$. Then we get $ura \in (P :_R M) \cap S$ which is a contradiction. Thus we obtain $(P :_R M) \cap S^* = \emptyset$. Then by (ii), P is an $S^\star$-2-absorbing submodule of M since $S \subseteq S^\star$. For the converse, suppose that P is an $S^\star$-2-absorbing submodule of M. Given $abm \in P$ for some $a, b \in R$ and $m \in M$. Then there exists $s' \in S^\star$ such that $s'ab \in (P :_R M)$ or $s'am \in P$ or $s'bm \in P$. On the other hand, $\frac{s'}{1}\frac{x}{s} = 1$ for some $x \in R, s \in S$ and so $us'x = us$ for some $u \in S$. Note that $t = us \in S$. Also, we have $tab \in (P :_R M)$ or $tam \in P$ or $tbm \in P$. Thus P is an S-2-absorbing submodule.

(iv): Let $\frac{a}{s}\frac{b}{t}\frac{m}{u} \in P$ for some $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ and $\frac{m}{u} \in S^{-1}M$. Then we get $u'abm = (u'a)bm \in P$ for some $u' \in S$. By the assumption, there is an $s' \in S$ such that $s'(u'a)b \in (P :_R M)$ or $s'(u'a)m \in P$ or $s'bm \in P$. Thus we have either $\frac{a}{s}\frac{b}{t} = \frac{s'u'ab}{s'u'st} \in S^{-1}(P :_R M) \subseteq (S^{-1}P :_{S^{-1}R} S^{-1}M)$ or $\frac{a}{s}\frac{m}{u} = \frac{s'u'am}{s'u'su} \in S^{-1}P$ or $\frac{b}{t}\frac{m}{u} = \frac{s'bm}{s'tu} \in S^{-1}P$. Hence, $S^{-1}P$ is a 2-absorbing submodule of $S^{-1}M$. $\square$

The converses of (i) in previous proposition is not true in general. To see this, take S as the set of units in a ring R and remember that any 2-absorbing submodule need not be a prime submodule. The following example shows that the converse of (iv) in previous proposition is not always true.

Example 4. *($S^{-1}P$ is 2-absorbing but P is not S-2-absorbing): Consider the* $\mathbb{Z}$*-module* $M = \mathbb{Q}^3$ *and* $S = reg(\mathbb{Z}) = \mathbb{Z} - \{0\}$*. Let* $P = \{(m, n, 0) : m, n \in \mathbb{Z}\}$*. Then note that* $(P :_{\mathbb{Z}} M) = 0$ *and* $(P :_{\mathbb{Z}} M) \cap S = \emptyset$*. Now, take* $s \in S$*. Then there exist prime numbers* $p \neq q$ *such that* $\gcd(p, s) = \gcd(q, s) = 1$*. Also it is clear that* $pq(\frac{1}{p}, \frac{1}{q}, 0) = (q, p, 0) \in P$ *but* $sp(\frac{1}{p}, \frac{1}{q}, 0) \notin P$*,* $sq(\frac{1}{p}, \frac{1}{q}, 0) \notin P$ *and* $spq \neq 0$*. Thus* P *is not* S*-2-absorbing submodule. Further, note that* $S^{-1}M$ *is a vector space and the proper subspace* $S^{-1}P$ *is a 2-absorbing submodule.*

Proposition 2. *Let* S *be a multiplicatively closed subset of* R *and* M *an* R*-module. Then the intersection of two* S*-prime submodule is an* S*-2-absorbing.*

Proof. Let P_1, P_2 be two S-prime submodules of M and $P = P_1 \cap P_2$. Let $abm \in P$ for some $a, b \in R$ and $m \in M$. Since P_1 is an S-prime submodule and $abm \in P_1$, there exists $s_1 \in S$ such that $s_1 a \in (P_1 :_R M)$ or $s_1 bm \in P_1$. Now, we will show that $s_1 bm \in P_1$ implies that $s_1 b \in (P_1 :_R M)$ or $s_1 m \in P_1$. Assume that $s_1 bm = b(s_1 m) \in P_1$. Since P_1 is an S-prime submodule, we get either $s_1 b \in (P_1 :_R M)$ or $s_1^2 m \in P_1$. If $s_1 b \in (P_1 :_R M)$, then we are done. So assume that $s_1^2 m \in P_1$. By [18, Lemma 2.16], we know that $(P_1 :_M s_1^2) \subseteq (P_1 :_M s_1)$ and this yields $m \in (P_1 :_M s_1^2) \subseteq (P_1 :_M s_1)$. Thus we have $s_1 m \in P_1$. In a similar manner, since P_2 is an S-prime submodule, there exists $s_2 \in S$ such that $s_2 a \in (P_2 :_R M)$ or $s_2 b \in (P_2 :_R M)$ or $s_2 m \in P_2$. Without loss of generality, we may assume that $s_1 a \in (P_1 :_R M)$ and $s_2 m \in P_2$. Now, put $s = s_1 s_2 \in S$. This implies that $sam \in P$ and hence P is an S-2-absorbing submodule of M. □

The intersection of two S-2-absorbing submodules is not necessarily S-2-absorbing. See the following example.

Example 5. *Let* p, q, r *be distinct prime numbers. Consider* $\mathbb{Z}$*-module* $M = \mathbb{Z} \times \mathbb{Z}$*. Now, set* $S = \{m \in \mathbb{Z} : \gcd(m, pqr) = 1\}$*. Then* S *is a multiplicatively closed subset of* $\mathbb{Z}$*. Then note that* $P_1 = pq\mathbb{Z} \times \mathbb{Z}$ *and* $P_2 = \mathbb{Z} \times pr\mathbb{Z}$ *are* S*-2-absorbing submodules of* M*. Also we have* $P = P_1 \cap P_2 = pq\mathbb{Z} \times pr\mathbb{Z}$*. Now, take* $s \in S$*. Then* $\gcd(s, p) = \gcd(s, q) = \gcd(s, r) = 1$*. Also note that* $pq(1, r) = (pq, pqr) \in P$ *but* $spq \notin (P :_{\mathbb{Z}} M)$*,* $sp(1, r) \notin P$ *and* $sq(1, r) \notin P$*. Thus* P *is not* S*-2-absorbing submodule.*

Lemma 1. *Let* S *be a multiplicatively closed subset of* R*. If* P *is a submodule of* M *with* $(P :_R M) \cap S = \emptyset$*, then the following statements are equivalent:*

(i) P *is an* S*-2-absorbing submodule of* M*.*

(ii) There exists an $s \in S$ *such that* $abN \subseteq P$ *for some* $a, b \in R$ *and a submodule* N *of* M *implies either* $saN \subseteq P$ *or* $sbN \subseteq P$ *or* $sab \in (P :_R M)$*.*

Proof. $(i) \Rightarrow (ii)$: Suppose that P is an S-2-absorbing submodule of M. We know that there exists $s \in S$ such that $xym \in P$ for some $x, y \in R$ and

$m \in M$ implies $sxy \in (P :_R M)$ or $sxm \in P$ or $sym \in P$. Let $abN \subseteq P$ for some $a, b \in R$ and a submodule N of M. Now, we will show that $saN \subseteq P$ or $sbN \subseteq P$ or $sab \in (P :_R M)$. Suppose to the contrary. Then $saN \nsubseteq P$, $sbN \nsubseteq P$ and $sab \notin (P :_R M)$. We have $n_1, n_2 \in N$ with $san_1 \notin P$ and $sbn_2 \notin P$. Since $abn_1 \in P$, $sab \notin (P :_R M)$ and $san_1 \notin P$, we get $sbn_1 \in P$ since P is S-2-absorbing. In a similar manner, we have $san_2 \in P$. On the other hand, we have $ab(n_1 + n_2) \in P$ and $sab \notin (P :_R M)$. Then either $sa(n_1 + n_2) \in P$ or $sb(n_1 + n_2) \in P$ since P is S-2-absorbing. Let $sa(n_1 + n_2) = san_1 + san_2 \in P$. Then this yields that $san_1 \in P$ since $san_2 \in P$, a contradiction. Let $sb(n_1 + n_2) = sbn_1 + sbn_2 \in P$. Then we get $sbn_2 \in P$ since $sbn_1 \in P$, again a contradiction. Therefore, $saN \subseteq P$ or $sbN \subseteq P$ or $sab \in (P :_R M)$.

$(ii) \Rightarrow (i)$: It is clear. $\square$

Corollary 1. *Let S be a multiplicatively closed subset of R and P an ideal of R with $P \cap S = \emptyset$. Then P is an S-2-absorbing ideal of R if and only if there exists an $s \in S$ such that $abI \subseteq P$ for some $a, b \in R$ and an ideal I of R implies either $saI \subseteq P$ or $sbI \subseteq P$ or $sab \in P$.*

Theorem 1. *Let S be a multiplicatively closed subset of R and P a submodule of R-module M with $(P :_R M) \cap S = \emptyset$. P is an S-2-absorbing submodule of M if and only if there exists an $s \in S$ such that $IJN \subseteq P$ for some ideals I, J of R and some submodule N of M implies either $sIN \subseteq P$ or $sJN \subseteq P$ or $sIJ \subseteq (P :_R M)$.*

Proof. $(\Leftarrow)$: Directly from definition.
$(\Rightarrow)$: Suppose that P is an S-2-absorbing submodule of M. Then there exists an $s \in S$ and whenever $abm \in P$ for some $a, b \in R$ and $m \in M$ then either $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. Given $IJN \subseteq P$ for some ideals I, J of R and some submodule N of M, we must show that $sIN \subseteq P$ or $sJN \subseteq P$ or $sIJ \subseteq (P :_R M)$. Suppose to the contrary. Then there exists $x \in I$ and $y \in J$ such that $sxN \nsubseteq P$ and $syN \nsubseteq P$. By Lemma 1, it is obtained that $sxy \in (P :_R M)$ since $xyN \subseteq P$. On the other hand, since $sIJ \nsubseteq (P :_R M)$, there exist $a \in I$ and $b \in J$ such that $sab \notin (P :_R M)$. Then by Lemma 1, we have $saN \subseteq P$ or $sbN \subseteq P$ since $abN \subseteq P$. Consider the following three cases:
First case: Assume that $saN \subseteq P$ but $sbN \nsubseteq P$. Note that $xbN \subseteq P$, $sbN \nsubseteq P$ and $sxN \nsubseteq P$. Then we conclude that $sxb \in (P :_R M)$ by Lemma 1. Since $sxN \nsubseteq P$ and $saN \subseteq P$, we also have $s(x + a)N \nsubseteq P$. Then by Lemma 1, we have $s(a+x)b \in (P :_R M)$ since $(a+x)bN \subseteq P$, $s(a+x)N \nsubseteq P$ and $sbN \nsubseteq P$. As $s(a + x)b \in (P :_R M)$ and $sxb \in (P :_R M)$, we get $sab \in (P :_R M)$ which is a contradiction.
Second case: Assume that $saN \nsubseteq P$ and $sbN \subseteq P$. It can be easily obtained

in a similar manner to the first case.

Third case: Assume that $saN \subseteq P$ and $sbN \subseteq P$. Then we get $s(b+y)N \nsubseteq P$ since $syN \nsubseteq P$ and $sbN \subseteq P$. Thus by Lemma 1, we get $sx(b+y) \in (P :_R M)$ since $x(b+y)N \subseteq P$, $s(b+y)N \nsubseteq P$ and $sxN \nsubseteq P$. As $sx(b+y) \in (P :_R M)$ and $sxy \in (P :_R M)$, we have $sxb \in (P :_R M)$. Also, it is clear that $s(a+x)N \nsubseteq P$ since $sxN \nsubseteq P$ and $saN \subseteq P$. Then by Lemma 1, $s(a+x)y \in (P :_R M)$ since $(a+x)yN \subseteq P$, $s(a+x)N \nsubseteq P$ and $syN \nsubseteq P$. As $s(a+x)y \in (P :_R M)$ and $sxy \in (P :_R M)$, we conclude that $say \in (P :_R M)$. Also, by Lemma 1, we get the result that $s(a+x)(b+y) = sab+say+sxb+sxy \in (P :_R M)$ since $(a+x)(b+y)N \subseteq P$, $s(a+x)N \nsubseteq P$ and $s(b+y)N \nsubseteq P$. As $s(a+x)(b+y) \in (P :_R M)$ and $say, sxy, sxb \in (P :_R M)$, we have $sab \in (P :_R M)$, again a contradiction. □

Corollary 2. *Let S be a multiplicatively closed subset of R and P an ideal of R with $P \cap S = \emptyset$. Then the following statements are equivalent:*

(i) P is an S-2-absorbing ideal of R.

(ii) There exists an $s \in S$ such that $IJK \subseteq P$ for some ideals I, J and K of R implies $sIJ \subseteq P$ or $sIK \subseteq P$ or $sJK \subseteq P$.

Let M be a multiplication R-module and K, L be two submodules of M. Then $K = IM$ and $L = JM$ for some ideals I, J of R. Also the product of K and L is defined as $KL = IJM$ [1]. Further, note that this product is independent of the presentations of submodules K and L of M [1, Theorem 3.4].

Proposition 3. *Let S be a multiplicatively closed subset of R. If P is an S-2-absorbing submodule of M, then $(P :_R M)$ is an S-2-absorbing ideal of R. Also, the converse is true in case M is a multiplication module.*

Proof. $(\Rightarrow)$: Assume that P is an S-2-absorbing submodule of M. Let $abc \in (P :_R M)$ for some $a, b, c \in R$. Then we have $RaRb(cM) \subseteq P$. Since P is an S-2-absorbing submodule, by Theorem 1, we have an fixed $s \in S$ such that $sRaRb \subseteq (P :_R M)$ or $sRa(cM) \subseteq P$ or $sRb(cM) \subseteq P$. Thus we get either $sab \in (P :_R M)$ or $sac \in (P :_R M)$ or $sbc \in (P :_R M)$. Hence, $(P :_R M)$ is an S-2-absorbing ideal of R.

$(\Leftarrow)$: Suppose that $(P :_R M)$ is an S-2-absorbing ideal of R. Let $IJN \subseteq P$ for some ideals I, J of R and some submodule N of M. Then it is easy to note that $IJ(N :_R M) \subseteq (IJN :_R M) \subseteq (P :_R M)$. By Corollary 2, there exists an $s \in S$ such that either $sIJ \subseteq (P :_R M)$ or $sI(N :_R M) \subseteq (P :_R M)$ or $sJ(N :_R M) \subseteq (P :_R M)$. Since M is multiplication, we have $sIJ \subseteq (P :_R M)$ or $sIN \subseteq P$ or $sJN \subseteq P$. Therefore, P is an S-2-absorbing submodule of M. □

As a result of Proposition 3 and Theorem 1, we give the following corollary.

Corollary 3. *Let M be a multiplication R-module, S be a multiplicatively closed subset of R and P a submodule of M with $(P :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(i) P is an S-2-absorbing submodule of M.

(ii) There exists an $s \in S$ such that $KLN \subseteq P$ for some submodules K, L, N of M implies $sKL \subseteq P$ or $sKN \subseteq P$ or $sLN \subseteq P$.

Theorem 2. *Let M be a finitely generated multiplication R-module and S be a multiplicatively closed subset of R. Suppose that P is a submodule of M with $(P :_R M) \cap S = \emptyset$. Then the following statements are equivalent:*

(i) P is an S-2-absorbing submodule of M.

(ii) $(P :_R M)$ is an S-2-absorbing ideal of R.

(iii) $P = IM$ for some S-2-absorbing ideal I of R with $ann(M) \subseteq I$.

Proof. $(i) \Leftrightarrow (ii)$: It is clear from Proposition 3.
$(ii) \Rightarrow (iii)$: It is obvious.

$(iii) \Rightarrow (i)$: Let $JKN \subseteq P$ for some ideals J, K of R and some submodule N of M. Then we have $JK(N :_R M)M \subseteq P = IM$. Also by [19, Theorem 9 Corollary], we get $JK(N :_R M) \subseteq I + ann(M) = I$. Then by Corollary 2, there is an $s \in S$ such that $sJK \subseteq I$ or $sJ(N :_R M) \subseteq I$ or $sK(N :_R M) \subseteq I$. So we conclude that $sJK \subseteq I \subseteq (P :_R M)$ or $sJ(N :_R M)M \subseteq IM = P$ or $sK(N :_R M)M \subseteq IM = P$, that is, $sJK \subseteq I \subseteq (P :_R M)$ or $sJN \subseteq P$ or $sKN \subseteq P$. Hence, P is an S-2-absorbing submodule of M. $\square$

Proposition 4. *Suppose that $f : M \to M'$ is an R-homomorphism and $S \subseteq R$ is a multiplicatively closed subset. The following statements hold:*

(i) If P' is an S-2-absorbing submodule of M' and $(f^{-1}(P') :_R M) \cap S = \emptyset$, then $f^{-1}(P')$ is an S-2-absorbing submodule of M.

(ii) If f is an epimorphism and P is an S-2-absorbing submodule of M containing $Ker(f)$, then $f(P)$ is an S-2-absorbing submodule of M'.

Proof. (i) Let $abm \in f^{-1}(P')$ for some $a, b \in R$ and $m \in M$. Then we get $f(abm) = abf(m) \in P'$. Since P' is an S-2-absorbing submodule, there exists $s \in S$ such that either $sab \in (P' :_R M')$ or $saf(m) = f(sam) \in P'$ or $sbf(m) = f(sbm) \in P'$. If $sab \in (P' :_R M')$, then we conclude that $sab \in (f^{-1}(P') :_R M)$ since $(P' :_R M') \subseteq (f^{-1}(P') :_R M)$. On the other hand, if $f(sam) \in P'$ or $f(sbm) \in P'$, we can conclude either $sam \in f^{-1}(P')$ or $sbm \in f^{-1}(P')$. Hence, $f^{-1}(P')$ is an S-2-absorbing submodule of M.

(ii) Suppose that P is an S-2-absorbing submodule of M containing $Ker(f)$. First, we will show that $(f(P) :_R M') \cap S = \emptyset$. Indeed, if $(f(P) :_R M') \cap S \neq \emptyset$, there is an $s \in S$ such that $s \in (f(P) :_R M')$. This implies that $sM' \subseteq f(P)$ and so $f(sM) = sf(M) \subseteq sM' \subseteq f(P)$. Thus, we have $sM \subseteq sM + Ker(f) \subseteq$

$P + Ker(f) = P$, that is, $sM \subseteq P$ which contradicts with P is an S-2-absorbing submodule. Now, suppose that $abm' \in f(P)$ for some $a, b \in R$ and $m' \in M'$. Then $m' = f(m)$ for some $m \in M$ as f is an epimoprhism. This implies that $abm' = abf(m) = f(abm) \in f(P)$. As $Ker(f) \subseteq P$, we get $abm \in P$. Since P is an S-2-absorbing submodule of M, there is an $s \in S$ such that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. Consequently, we obtain that $sab \in (f(P) :_R M')$ or $f(sam) = sf(am) = sam' \in f(P)$ or $f(sbm) = sf(bm) = sbm' \in f(P)$ since $(P :_R M) \subseteq (f(P) :_R M')$. Therefore, $f(P)$ is an S-2-absorbing submodule of M'. $\square$

Corollary 4. *Let L be a submodule of an R-module M and $S \subseteq R$ be a multiplicatively closed subset. The following statements hold:*

(i) If P' is an S-2-absorbing submodule of M with $(P' :_R L) \cap S = \emptyset$, then $L \cap P'$ is an S-2-absorbing submodule of L.

(ii) Assume that P is a submodule of M containing L. Then P is an S-2-absorbing submodule of M if and only if P/L is an S-2-absorbing submodule of M/L.

Proof. (i) Consider that the injection $i : L \to M$ defined by $i(m) = m$ for all $m \in L$. Then we have $i^{-1}(P') = L \cap P'$. Now, we will show that $(i^{-1}(P') :_R L) \cap S = \emptyset$. Indeed, if $s \in (i^{-1}(P') :_R L) \cap S$, then we have $sL \subseteq i^{-1}(P') = L \cap P' \subseteq P'$ and so $s \in (P' :_R L) \cap S$, a contradiction. The rest is obtained by Proposition 4 (1).

(ii) ($\Rightarrow$): Consider the canonical homomorphism $\pi : M \to M/L$ defined by $\pi(m) = m + L$ for all $m \in M$. The rest of proof is clear by Proposition 4 (2). ($\Leftarrow$): Let $abm \in P$ for some $a, b \in R$ and $m \in M$. Then we have $ab(m+L) = abm + L \in P/L$. Thus, there exists an $s \in S$ such that $sab \in (P/L :_R M/L) = (P :_R M)$ or $sa(m+L) = sam + L \in P/L$ or $sb(m+L) = sbm + L \in P/L$ by the assumption. Therefore, we get $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. Consequently, P is an S-2-absorbing submodule of M. $\square$

Proposition 5. *Let S be a multiplicatively closed subset of R and P be an S-2-absorbing submodule of R-module M. Suppose that $(P :_R m) \cap S = \emptyset$ for all $m \in M - P$. If $(P :_R M)$ is an S-prime ideal of R, then $(P :_R m)$ is an S-prime ideal of R for each $m \in M - P$.*

Proof. Let P be an S-2-absorbing submodule of R-module M and $(P :_R m) \cap S = \emptyset$ for all $m \in M$. First note that $(P :_R M) \cap S = \emptyset$ since $(P :_R M) \subseteq (P :_R m)$ for each $m \in M$. Assume that $(P :_R M)$ is an S-prime ideal of R. Given $ab \in (P :_R m)$ for some $a, b \in R$, we must show that there exists a fixed $s \in S$ such that either $sa \in (P :_R m)$ or $sb \in (P :_R m)$. Then $abm \in P$. Since P is an S-2-absorbing submodule, there exists a fixed $s_1 \in S$ such that $xym^\star \in P$ for some $x, y \in R$ and $m^\star \in M$ implies that either $s_1xy \in (P :_R M)$

or $s_1xm^\star \in P$ or $s_1ym^\star \in P$. On the other hand, since $(P :_R M)$ is an S-prime ideal of R, there exists a fixed $s_2 \in S$ such that $xy \in (P :_R M)$ for some $x, y \in R$ implies either $s_2x \in (P :_R M)$ or $s_2y \in (P :_R M)$. Now, put $s = s_1s_2 \in S$. Then we can easily get $s_1ab \in (P :_R M)$ or $s_1am \in P$ or $s_1bm \in P$ since P is an S-2-absorbing and $abm \in P$. If $s_1am \in P$ or $s_1bm \in P$, then $sa \in (P :_R m)$ or $sb \in (P :_R M)$. Assume that $s_1ab \in (P :_R M)$. Since $(P :_R M)$ is S-prime ideal, then we get either $s_1s_2a \in (P :_R M)$ or $s_2b \in (P :_R M)$. Thus we conclude $sa \in (P :_R M) \subseteq (P :_R m)$ or $sb \in (P :_R M) \subseteq (P :_R m)$. □

Assume that M is an R-module. The trivial extension $R \propto M = R \oplus M$ of M is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, m') = (ab, am' + bm)$ for each $a, b \in R$ and $m, m' \in M$ [15]. Let I be an ideal of R and N a submodule of M. Then $I \propto N$ is an ideal of $R \propto M$ if and only if $IM \subseteq N$ [3, Theorem 3.3]. In this case, $I \propto N$ is called a homogeneous ideal of $R \propto M$. Also, if S is a multiplicatively closed subset of R and P is a submodule of M, then $S \propto P = \{(s, p) : s \in S, p \in P\}$ is a multiplicatively closed subset of $R \propto M$ [3, Theorem 3.8].

Proposition 6. *Suppose that S is a multiplicatively closed subset of R and P is an ideal of R with $P \cap S = \emptyset$. Then the following statements are equivalent:*
(i) P is an S-2-absorbing ideal of R.
(ii) $P \propto M$ is an $S \propto 0$-2-absorbing ideal of $R \propto M$.
(iii) $P \propto M$ is an $S \propto M$-2-absorbing ideal of $R \propto M$.

Proof. $(i) \Rightarrow (ii)$: Let $(x, m)(y, m')(z, m^\star) = (xyz, xym^\star + xzm' + yzm) \in P \propto M$ for some $x, y, z \in R$ and $m, m', m^\star \in M$. Then we get $xyz \in P$. By the assumption, we have an $s \in S$ such that $sxy \in P$ or $sxz \in P$ or $syz \in P$. Then we obtain $(s, 0)(x, m)(y, m') = (sxy, sxm' + sym) \in P \propto M$ or $(s, 0)(x, m)(z, m^\star) = (sxz, sxm^\star + szm) \in P \propto M$ or $(s, 0)(y, m')(z, m^\star) = (syz, sym^\star + szm') \in P \propto M$, where $(s, 0) \in S \propto 0$. Thus, $P \propto M$ is an $S \propto 0$-2-absorbing ideal of $R \propto M$.
$(ii) \Rightarrow (iii)$: It is clear from Proposition 3 since $S \propto 0 \subseteq S \propto M$.
$(iii) \Rightarrow (i)$: Assume that $xyz \in P$ for some $x, y, z \in R$. Then $(x, 0)(y, 0)(z, 0) \in P \propto M$. Since $P \propto M$ is an $S \propto M$-2-absorbing ideal of $R \propto M$, there is an $(s, m) \in S \propto M$ such that $(s, m)(x, 0)(y, 0) = (sxy, xym) \in P \propto M$ or $(s, m)(y, 0)(z, 0) = (syz, yzm) \in P \propto M$ or $(s, m)(x, 0)(z, 0) = (sxz, xzm) \in P \propto M$ and hence we get $sxy \in P$ or $syz \in P$ or $sxz \in P$. Therefore, P is an S-2-absorbing ideal of R. □

Let M_i be an R_i-module for each $i = 1, 2, \ldots, n$ and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with componentwise addition and scalar multiplication. Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed subset of R. Furthermore, each submodule N of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i. Now, we determine S-2-absorbing submodules of cartesian product of modules.

Proposition 7. *Let M_i be an R_i-module and S_i be a multiplicatively closed subset of R_i for each $i = 1, 2$. Let $M = M_1 \times M_2$, $R = R_1 \times R_2$ and $S = S_1 \times S_2$. Assume that N_1 is a submodule of M_1, N_2 is a submodule of M_2 and $N = N_1 \times N_2$. Then the following statements are equivalent:*

(i) N is an S-2-absorbing submodule of M.

(ii) $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and N_2 is an S_2-2-absorbing submodule of M_2 or N_1 is an S_1-2-absorbing submodule of M_1 and $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ or N_1 is an S_1-prime submodule of M_1 and N_2 is an S_2-prime submodule of M_2.

Proof. $(i) \Rightarrow (ii)$: Assume that N is an S-2-absorbing submodule of M. First, note that $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ is an S-2-absorbing ideal of R by Proposition 3. So that $(N_1 :_{R_1} M_1) \cap S_1 = \emptyset$ or $(N_2 :_{R_2} M_2) \cap S_2 = \emptyset$. Assume that $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$. Now, we will show that N_2 is an S_2-2-absorbing submodule of M_2. Let $xym \in N_2$ for some $x, y \in R_2$ and $m \in M_2$. Then we have $(0_{R_1}, x)(0_{R_1}, y)(0_{M_1}, m) = (0_{M_1}, xym) \in N_1 \times N_2 = N$. As N is an S-2-absorbing submodule of M, there exists $s = (s_1, s_2) \in S$ such that $s(0_{R_1}, x)(0_{R_1}, y) = (0_{R_1}, s_2xy) \in (N :_R M)$ or $s(0_{R_1}, x)(0_{M_1}, m) = (0_{M_1}, s_2xm) \in N$ or $s(0_{R_1}, y)(0_{M_1}, m) = (0_{M_1}, s_2ym) \in N$. This implies that $s_2xy \in (N_2 :_{R_2} M_2)$ or $s_2xm \in N_2$ or $s_2ym \in N_2$. Hence, N_2 is an S_2-2-absorbing submodule of M_2. If $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$, similarly N_1 is an S_1-2-absorbing submodule of M_1. Now assume that $(N_1 :_{R_1} M_1) \cap S_1 = \emptyset$ and $(N_2 :_{R_2} M_2) \cap S_2 = \emptyset$. We will show that N_1 is an S_1-prime submodule of M_1 and N_2 is an S_2-prime submodule of M_2. First, note that there exists a fixed $s = (s_1, s_2) \in S$ satisfying N to be an S-2-absorbing submodule of M. Suppose that N_1 is not an S_1-prime submodule of M_1. Then there exists $a \in R_1$ and $m_1 \in M_1$ such that $am_1 \in N_1$ but $s_1a \notin (N_1 :_{R_1} M_1)$ and $s_1m_1 \notin N_1$. On the other hand $(N_2 :_{R_2} M_2) \cap S_2 = \emptyset$, $s_2 \notin (N_2 :_{R_2} M_2)$ so there exists $m_2 \in M_2$ such that $s_2m_2 \notin N_2$. Also note that $(a, 1)(1, 0)(m_1, m_2) = (am_1, 0_{M_2}) \in N_1 \times N_2 = N$. Since N is an S-2-absorbing submodule of M, we have either $(s_1, s_2)(a, 1)(1, 0) = (s_1a, 0) \in (N :_R M)$ or $(s_1, s_2)(a, 1)(m_1, m_2) = (s_1am_1, s_2m_2) \in N$ or $(s_1, s_2)(1, 0)(m_1, m_2) = (s_1m_1, 0_{M_2}) \in N$. Then we conclude that either $s_1a \in (N_1 :_{R_1} M_1)$ or $s_1m_1 \in N_1$ or $s_2m_2 \in N_2$ which both them are contradictions. Hence, N_1 is an S_1-prime submodule of M_1. Similar argument shows that N_2 is an S_2-prime submodule of M_2.

$(ii) \Rightarrow (i)$: Let $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$ and N_2 be an S_2-2-absorbing submodule of M_2. Now, we will show that N is an S-2-absorbing submodule of M. First, note that $(N :_R M) \cap S = \emptyset$. Let $a, b \in R_1$; $x, y \in R_2$; $m_1 \in M_1$ and $m_2 \in M_2$ such that $(a, x)(b, y)(m_1, m_2) = (abm_1, xym_2) \in N$. As $(N_1 :_{R_1} M_1) \cap S_1 \neq \emptyset$, there exists $s_1 \in S_1$ such that $s_1 m \in N_1$ for all $m \in M_1$. Also there exists a fixed $s_2 \in S_2$ satisfying N_2 to be an S_2-2-absorbing submodule of M_2. Now, put $s = (s_1, s_2) \in S$. Also note that $xym_2 \in N_2$. Since N_2 is an S_2-2-absorbing submodule of M_2, we conclude either $s_2xy \in (N_2 :_{R_2} M_2)$ or $s_2xm_2 \in N_2$ or $s_2ym_2 \in N_2$. This yields that $s(a, x)(b, y) = (s_1ab, s_2xy) \in (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2) = (N :_R M)$ or $s(a, x)(m_1, m_2) = (s_1am_1, s_2xm_2) \in N_1 \times N_2 = N$ or $s(b, y)(m_1, m_2) = (s_1bm_1, s_2ym_2) \in N_1 \times N_2 = N$. Hence, we conclude that N is an S-2-absorbing submodule of M. If $(N_2 :_{R_2} M_2) \cap S_2 \neq \emptyset$ and N_1 is an S_1-2-absorbing submodule of M_1, similar argument shows that N is an S-2-absorbing submodule of M. Now assume that N_1 is an S_1-prime submodule of M_1 and N_2 is an S_2-prime submodule of M_2. Let $a, b \in R_1$; $x, y \in R_2$; $m_1 \in M_1$ and $m_2 \in M_2$ such that $(a, x)(b, y)(m_1, m_2) = (abm_1, xym_2) \in N$. Then we have $abm_1 \in N_1$ and $xym_2 \in N_2$. Since N_1 is an S_1-prime submodule of M_1, there exists a fixed $s_1 \in S_1$ such that either $s_1a \in (N_1 :_{R_1} M_1)$ or $s_1b \in (N_1 :_{R_1} M_1)$ or $s_1m_1 \in N_1$. Similarly, there exists $s_2 \in S_2$ such that either $s_2x \in (N_2 :_{R_2} M_2)$ or $s_2y \in (N_2 :_{R_2} M_2)$ or $s_2m_2 \in N_2$. Put $s = (s_1, s_2) \in S$. Also without loss of generality, we may assume that $s_1a \in (N_1 :_{R_1} M_1)$ and $s_2m_2 \in N_2$. Then we have $s(a, x)(m_1, m_2) = (s_1am_1, s_2xm_2) \in N_1 \times N_2 = N$. Hence, N is an S-2-absorbing submodule of M. $\square$

As seen in the above theorem, we have that if N_1 is an S_1-2-absorbing submodule of M_1 and N_2 is an S_2-2-absorbing submodule of M_2, then $N_1 \times N_2$ may not be an $S_1 \times S_2$-2-absorbing submodule of $M_1 \times M_2$. See the following example.

Example 6. *Consider the submodules $N_1 = 9\mathbb{Z}$ and $N_2 = 4\mathbb{Z}$ of $\mathbb{Z}$-module $\mathbb{Z}$. Let $S_1 = \mathbb{Z} - 3\mathbb{Z}$ and $S_2 = \mathbb{Z} - 2\mathbb{Z}$. Note that N_1 and N_2 are S_1 and S_2-2-absorbing submodules of $\mathbb{Z}$, respectively. But $N = N_1 \times N_2$ is not an $S = S_1 \times S_2$-2-absorbing submodule of $M = \mathbb{Z} \times \mathbb{Z}$ since $(3, 2)(1, 2)(3, 1) \in N$ but for each $s = (s_1, s_2) \in S$, $s(3, 2)(1, 2) \notin (N :_{\mathbb{Z}} M)$, $s(1, 2)(3, 1) \notin N$ and $s(3, 2)(3, 1) \notin N$.*

Theorem 3. *Let $n \geq 1$ and M_i be an R_i-module and S_i be a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$. Let $M = M_1 \times M_2 \times \cdots \times M_n$, $R = R_1 \times R_2 \times \cdots \times R_n$ and $S = S_1 \times S_2 \times \cdots \times S_n$. Assume that N_i is a submodule of M_i for each $i = 1, 2, \ldots, n$ and $N = N_1 \times N_2 \times \cdots \times N_n$. Then the following statements are equivalent:*

(i) N is an S-2-absorbing submodule of M.

(ii) N_k *is an* S_k*-2-absorbing submodule of* M_k *for some* $1 \leq k \leq n$ *and* $(N_t :_{R_t} M_t) \cap S_t \neq \emptyset$ *for each* $t \in \{1, 2, \ldots, n\} - \{k\}$ *or* N_{k_1} *is an* S_{k_1}*-prime submodule of* M_{k_1} *and* N_{k_2} *is an* S_{k_2}*-prime submodule of* M_{k_2} *for some* $1 \leq k_1 \neq k_2 \leq n$ *and* $(N_t :_{R_t} M_t) \cap S_t \neq \emptyset$ *for each* $t \in \{1, 2, \ldots, n\} - \{k_1, k_2\}$.

Proof. To prove the result, we will use induction on n. The implication $(i) \Leftrightarrow (ii)$ is trivial when $n = 1$. If $n = 2$, $(i) \Leftrightarrow (ii)$ follows from Proposition 7. Assume that the implication $(i) \Leftrightarrow (ii)$ is true for all $k < n$. Now, we will prove that $(i) \Leftrightarrow (ii)$ is true for $k = n$. Put $R = R' \times R_n$, $M = M' \times M_n$ and $S = S' \times S_n$, where $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$, $M' = M_1 \times M_2 \times \cdots \times M_{n-1}$ and $S' = S_1 \times S_2 \times \cdots \times S_{n-1}$. Also put $N = N' \times N_n$, where $N' = N_1 \times N_2 \times \cdots \times N_{n-1}$. Then by Proposition 7, N is an S-2-absorbing submodule of M if and only if $(N' :_{R'} M') \cap S' \neq \emptyset$ and N_n is an S_n-2-absorbing submodule of M_n or N' is an S'-2-absorbing submodule of M' and $(N_n :_{R_n} M_n) \cap S_n \neq \emptyset$ or N' is an S'-prime submodule of M' and N_n is an S_n-prime submodule of M_n. The rest follows from induction hypothesis and [18, Theorem 2.15]. □

Lemma 2. *Let* M *be an* R*-module and* $S \subseteq R$ *a multiplicatively closed subset of* R*. Assume that* P *is an* S*-2-absorbing submodule of* M*. Then the following statements are satisfied:*

(i) There exists $s \in S$ *such that* $(P :_M s^3) = (P :_M s^n)$ *for all* $n \geq 3$.

(ii) There exists $s \in S$ *such that* $(P :_R s^3 M) = (P :_R s^n M)$ *for all* $n \geq 3$.

Proof. (i) : Assume that P is an S-2-absorbing submodule of M. Then there exists $s \in S$ such that whenever $abm \in P$, where $a, b \in R$ and $m \in M$, then either $sab \in (P :_R M)$ or $sam \in N$ or $sbm \in P$. Now, take $m' \in (P :_M s^4)$. Then we have $s^4 m' = s^2 s^2 m' \in P$. As P is an S-2-absorbing submodule of M, we conclude that $s(s^2 m') = s^3 m' \in P$ and so $m' \in (P :_M s^3)$. Thus $(P :_M s^4) \subseteq (P :_M s^3)$. As the reverse inclusion always holds, we have $(P :_M s^4) = (P :_M s^3)$. Assume that $(P :_M s^3) = (P :_M s^k)$ for all $k < n$. Now, we will show that $(P :_M s^3) = (P :_M s^n)$. Let $m' \in (P :_M s^n)$. Then $s^n m' = s^2 (s^{n-2}) m' \in P$. As P is an S-2-absorbing submodule of M, we conclude either $s^3 m' \in P$ or $s^{n-1} m' \in P$. This implies that $m' \in (P :_M s^3) \cup (P :_M s^{n-1}) = (P :_M s^3)$ by induction hypothesis. Thus we have $(P :_M s^3) = (P :_M s^n)$.

(ii) : Follows from (i). □

Theorem 4. *Let* M *be an* R*-module and* $S \subseteq R$ *a multiplicatively closed subset of* R*. Assume that* P *is a submodule of* M *with* $(P :_R M) \cap S = \emptyset$*. Then the following statements are equivalent:*

(i) P *is an* S*-2-absorbing submodule.*

(ii) $(P :_M s)$ *is a 2-absorbing submodule for some* $s \in S$.

Proof. $(ii) \Rightarrow (i)$: Assume that $(P :_M s)$ is a 2-absorbing submodule for some $s \in S$. Let $abm \in P \subseteq (P :_M s)$ for some $a, b \in R$ and $m \in M$. As $(P :_M s)$ is a 2-absorbing submodule, we conlcude either $ab \in ((P :_M s) : M)$ or $am \in (P :_M s)$ or $bm \in (P :_M s)$. This implies that $sab \in (P :_R M)$ or $sam \in P$ or $sbm \in P$. Thus, P is an S-2-absorbing submodule.

$(i) \Rightarrow (ii)$: Let P be an S-2-absorbing submodule. Fix $s \in S$ satisfying P to be an S-2-absorbing submodule. Then by Lemma 2, we have $(P :_M s^3) = (P :_M s^n)$ and $(P :_R s^3M) = (P :_R s^nM)$ for all $n \geq 3$. Now, we will show that $(P :_M s^6) = (P :_M s^3)$ is a 2-absorbing submodule of M. Let $abm \in (P :_M s^6)$ for some $a, b \in R$ and $m \in M$. Then we have $s^6(abm) = (s^2a)(s^2b)(s^2m) \in P$. As P is an S-2-absorbing submodule, we conclude either $s(s^2a)(s^2b) = s^5ab \in (P :_R M)$ or $s(s^2a)(s^2m) = s^5am \in P$ or $s(s^2b)(s^2m) = s^5bm \in P$. This implies that $ab \in (P :_R s^5M) = (P :_R s^6M) = ((P :_M s^6) : M)$ or $am \in (P :_M s^5) = (P :_M s^6)$ or $bm \in (P :_M s^5) = (P :_M s^6)$. Hence, $(P :_M s^6)$ is a 2-absorbing submodule of M. $\square$

3 vn-regular modules

In this section, we will study the concept of vn-regular modules and characterize them in terms of S-2-absorbing submodules. Recall that M is said to be a vn-regular module if for each $m \in M$, there exists $a \in R$ such that $Rm = aM = a^2M$ [9]. It is clear that vn-regular modules are multiplication. To see this, take a vn-regular R-module M. Let N be a submodule of M. Then $N = \sum\limits_{n \in N} Rn$. Since M is vn-regular $Rn = a_nM = a_n^2M$ for some $a_n \in R$. Then we have

$$N = \sum_{n \in N} Rn = \sum_{n \in N} a_nM = \left(\sum_{n \in N} a_n\right) M.$$

Hence, M is a multiplication module. Also, if M is finitely generated vn-regular module, then $IM \cap JM = IJM$ for every ideal I and J of R [9, Lemma 6 and Theorem 1].

Proposition 8. *Let M be a finitely generated vn-regular module and $S \subseteq R$ a multiplicatively closed subset. Suppose that P is a submodule of M with $(P :_R M) \cap S = \emptyset$. Then P is an S-2-absorbing submodule of M if and only if there exists an $s \in S$ such that $K \cap L \cap N \subseteq P$ for some submodules K, L and N of M implies either $s(K \cap L) \subseteq P$ or $s(K \cap N) \subseteq P$ or $s(L \cap N) \subseteq P$.*

Proof. $(\Rightarrow)$: Let M be a finitely generated vn-regular module and P be an S-2-absorbing submodule of M, where S is a multiplicatively closed subset of R. Then M is multiplication. Assume that $K \cap L \cap N \subseteq P$ for some submodules K, L and N of M. This implies that $KLN = (K :_R M)(L :_R M)(N :_R$

$M)M \subseteq K \cap L \cap N \subseteq P$. By Corollary 3, there exists an $s \in S$ such that $sKL \subseteq P$ or $sKN \subseteq P$ or $sLN \subseteq P$. Since M is a finitely generated vn-regular module, by [9, Lemma 6 and Theorem 1], for any submodule N and N' of M we have $NN' = (N :_R M)(N' :_R M)M = (N :_R M)M \cap (N' :_R M)M = N \cap N'$. Then we conclude that $s(K \cap L) \subseteq P$ or $s(K \cap N) \subseteq P$ or $s(L \cap N) \subseteq P$.
($\Leftarrow$): Similar argument. $\square$

Let M be an R-module. $\mathcal{L}(M)$ be the lattice of all submodules of M and $\mathcal{L}(R)$ be the lattice of all ideals of R. Consider the mapping $\mu : \mathcal{L}(M) \to \mathcal{L}(R)$ defined by $\mu(N) = (N :_R M)$ for all $N \in \mathcal{L}(M)$. Recall from [20], an R-module M is said to be a *μ-module* if μ is an homomorphism. Smith, in [20, Lemma 3.1], showed that an R-module M is a μ-module if and only if $(N :_R M) + (K :_R M) = (N + K :_R M)$ for all $N, K \in \mathcal{L}(M)$. Also, the author showed that a finitely generated module is a μ-module if and only if it is multiplication [20, Theorem 3.8]. Now, in the following, we prove that Chinese remainder theorem for μ-modules.

Theorem 5. *(Chinese Remainder Theorem) Let M be a μ-module and K, N be two submodules such that $K + N = M$. Then $M/(K \cap N) \cong M/K \times M/N$.*

Proof. Suppose that M is a μ-module and K, N are two submodules such that $K + N = M$. Then by [20, Lemma 3.1], $(M :_R M) = R = (K :_R M) + (N :_R M)$. Then there exist $x \in (K :_R M)$ and $y \in (N :_R M)$ such that $1 = x + y$. Now, consider the mapping $\pi : M \to M/K \times M/N$ defined by $\pi(m) = (m + K, m + N)$ for each $m \in M$. Then π is well defined and R-homomorphism. Take $(m + K, m' + N) \in M/K \times M/N$ for some $m, m' \in M$. Then $m = xm + ym$ and $m' = xm' + ym'$. Now, put $m^\star = ym + xm'$. Then note that $m^\star + K = (ym + xm') + K = ym + K = (xm + ym) + K = m + K$ and similarly we have $m^\star + N = m' + N$. Thus $\pi(m^\star) = (m + K, m' + N)$ and so π is epimorphism. On the other hand $Ker(\pi) = K \cap N$ and so by the first isomorphism theorem, we get $M/(K \cap N) \cong M/K \times M/N$. $\square$

Recall from [11], an R-module M is said to be a *reduced module* if for each $a \in R$ and $m \in M$, $am = 0$ implies that $aM \cap Rm = 0$. Note that an R-module M is a reduced module if and only if $a^2m = 0$ for some $a \in R$ and $m \in M$ implies $am = 0$. In [9, Lemma 10], the authors proved that all finitely generated vn-regular modules are reduced.

Proposition 9. *Let M be a finitely generated reduced module and $S \subseteq R$ a multiplicatively closed subset. Suppose that all proper submodules are S-2-absorbing. Then for each $a \in R$, there exists $s^\star \in S$ such that $s^\star aM \subseteq a^2M$.*

Proof. Suppose that M is a finitely generated reduced module and its all proper submodules are S-2-absorbing, where S is a multiplicatively closed subset of R. Take an element $a \in R$. If $a^3M = M$, then $aM = M \subseteq a^2M = M$ and we are done. Assume that a^3M is a proper submodule. Since $a^3M = aa(aM) \subseteq a^3M$, by Lemma 1, there exists $s \in S$ such that $sa^2M \subseteq a^3M$. On the other hand, since M is finitely generated, we can write $M = Rm_1 + Rm_2 + \cdots + Rm_n$ for some $m_1, m_2, \ldots, m_n \in M$. Then for each $i = 1, 2, \ldots, n$, we have $sa^2m_i = a^3r_{i,1}m_1 + a^3r_{i,2}m_2 + \cdots + a^3r_{i,n}m_n$ and so we have $-a^3r_{i,1}m_1 - a^3r_{i,2}m_2 - \ldots + (sa^2 - a^3r_{i,i})m_i - \ldots - a^3r_{i,n}m_n = 0$. Thus we conclude that $\det(\Delta)M = 0$, where Δ is the n by n matrix

$$\begin{bmatrix} sa^2 - a^3r_{1,1} & -a^3r_{1,2} & \cdots & -a^3r_{1,n} \\ -a^3r_{2,1} & sa^2 - a^3r_{2,2} & \cdots & -a^3r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ -a^3r_{n,1} & -a^3r_{n,2} & \cdots & sa^2 - a^3r_{n,n} \end{bmatrix}_{n\times n}$$

. Then we conclude that $(a^2)^n(s^n + ax)M = 0$ for some $x \in R$. This implies that $(a(s^n + ax))^{2n} m = 0$ for all $m \in M$. Since M is reduced, we get $a(s^n + ax)m = 0$ and so $s^nam = -a^2xm$. This implies that $s^\star aM \subseteq a^2M$ where $s^\star = s^n \in S$. $\square$

Let M be an R-module. Recall from [2], M is said to be a *simple module* if the only submodules of M are $\{0\}$ and M. Note that all simple modules are vn-regular (See, [9, Example 2]). Also, an R-module M is said to be *semisimple* if $M = \oplus_{i\in I}M_i$ is the direct sum of simple submodules $\{M_i\}_{i\in I}$. Now, we characterize vn-regular modules in terms of 2-absorbing submodules.

Theorem 6. *Let M be a finitely generated module. The following staments are equivalent:*

(i) M is reduced multiplication module and all proper submodules are 2-absorbing.

(ii) M is vn-regular module and there are at most two prime submodules.

(iii) M is simple or M is multiplication module such that $M \cong M_1 \oplus M_2$ for some simple modules M_1 and M_2.

Proof. $(i) \Rightarrow (ii)$: Suppose that M is a finitely generated reduced multiplication module and all proper submodules are 2-absorbing. Put $S = u(R)$ and apply Proposition 9. Then $aM = a^2M$ for each $a \in R$, that is, a is M-vn-regular element of R. By [9, Theorem 1], $R/ann(M)$ is a vn-regular ring and so by [9, Lemma 7], M is vn-regular module. First, we will show that all prime submodules are maximal. Let $P^\star$ be a prime submodule of M. Then $(P^\star :_R M)$ is a prime ideal in R and $(P^\star :_R M)/ann(M)$ is a prime ideal in $R/ann(M)$. Since $R/ann(M)$ is vn-regular, $(P^\star :_R M)/ann(M)$ is a maximal ideal in $R/ann(M)$ and so $(P^\star :_R M)$ is a maximal ideal in R. Let N be a submodule of M containing $P^\star$. Since $(P^\star :_R M) \subseteq (N :_R M)$, we conclude either

$(N :_R M) = R$ or $(N :_R M) = (P^\star :_R M)$. As M is multiplication, $N = M$ or $N = P^\star$. Hence, $P^\star$ is a maximal submodule of M. Now, we will show that M has at most two prime submodules. Let P_1, P_2, P_3 be distinct prime submodules of M. Put $P = \bigcap_{i=1}^{3} P_i$. Then by assumption, P is 2-absorbing submodule. By [5, Proposition 1], $(P :_R M) = \bigcap_{i=1}^{3} (P_i :_R M)$ is a 2-absorbing ideal of R. As $\prod_{i=1}^{3} (P_i :_R M) \subseteq (P :_R M)$, by [4, Theorem 2.13], without loss of generality we may assume that $(P_1 :_R M)(P_2 :_R M) \subseteq (P :_R M) \subseteq (P_3 :_R M)$. As $(P_3 :_R M)$ is a prime ideal, we get either $(P_1 :_R M) \subseteq (P_3 :_R M)$ or $(P_2 :_R M) \subseteq (P_3 :_R M)$. Then we have $P_1 \subseteq P_3$ or $P_2 \subseteq P_3$. As P_1 and P_2 are maximal submodules, we have either $P_1 = P_3$ or $P_2 = P_3$. Hence, M has at most two prime submodules.

$(ii) \Rightarrow (iii)$: Suppose that M is finitely generated vn-regular module and M has at most two prime submodule. Since M is finitely generated vn-regular module, by [9, Lemma 6], $R/ann(M)$ is a vn-regular ring. Thus similar in the proof $(i) \Rightarrow (ii)$ shows that all prime submodules are maximal. Also by [9, Lemma 11], $0 = P$ or $0 = P \cap Q$, where P, Q are prime submodules of M. If $0 = P$, then M is simple. So assume that $0 = P \cap Q$ for some prime submodules P and Q of M. Then $P + Q = M$. By Theorem 5, $M \cong M/P \times M/Q$, where M/P and M/Q are simple modules.

$(iii) \Rightarrow (i)$: Suppose that M is simple. Then zero submodule is clearly prime so that 2-absorbing. Now, assume that $M \cong M_1 \oplus M_2$ is multiplication module for some simple modules M_1 and M_2. Without loss of generality, we may assume that $M = M_1 \oplus M_2$ is multiplication module, where M_1, M_2 are simple submodules of M. Since $M = M_1 \oplus M_2$ is finitely generated multiplication, M is μ-module so that $(M :_R M) = R = (M_1 + M_2 :_R M) = (M_1 :_R M) + (M_2 :_R M)$. This implies that $ann(M_1) + ann(M_2) = R$. Let K be a submodule of M. Then by [6, Lemma 2.6], all the posibilities are $0 \oplus 0, 0 \oplus M_2, M_1 \oplus 0, M_1 \oplus M_2$ for K. If K is $0 \oplus M_2$ or $M_1 \oplus 0$, then K is prime. Assume that $K = 0 \oplus 0$. Then $(K :_R M) = ann(M_1) \cap ann(M_2)$, where $ann(M_1)$ and $ann(M_2)$ are maximal ideals of R. Since M is multiplication module, by [4, Theorem 2.13] and [17, Theorem 2.3], K is a 2-absorbing submodule of M. Hence, all proper submodules are 2-absorbing. $\square$

References

[1] Ameri, R., On the prime submodules of multiplication modules, Int. J. Math. Sci. 2003(27) (2003), 1715-1724.

[2] Anderson, F. W., & Fuller, K. R., Rings and categories of modules (Vol. 13). Springer Science & Business Media, 2012.

[3] Anderson, D. D. and Winders, M., Idealization of a module. J. Comm. Alg. 1(1) (2009), 3-56.

[4] Badawi, A., On 2-absorbing ideals of commutative rings. Bulletin of the Australian Mathematical Society, 75(3) (2007), 417-429.

[5] Darani, A. Y., & Soheilnia, F., On n-absorbing submodules. Mathematical Communications, 17(2) (2012), 547-557.

[6] Bilgin, Z., Oral, K. H., Coprimely Structured Modules. Palestine Journal of Mathematics, 7(Special Issue I) (2018), 161-169.

[7] Darani, A. Y., Soheilnia, F., Tekir, U., & Ulucak, G., On weakly 2-absorbing primary submodules of modules over commutative rings. J. Korean Math. Soc, 54(5) (2017), 1505-1519.

[8] El-Bast, Z. A., & Smith, P. F., Multiplication modules. Communications in Algebra, 16(4) (1988), 755-779.

[9] Jayaram, C., & Tekir, Ü., von Neumann regular modules. Communications in Algebra, 46(5) (2018), 2205-2217.

[10] Koc, S., Uregen, R. N., & Tekir, U., On 2-Absorbing Quasi Primary Submodules. Filomat, 31(10) (2017), 2943-2950.

[11] Lee, T. K., & Zhou, Y., Reduced modules. Rings, modules, algebras and abelian groups, 236 (2004), 365-377.

[12] Lee, T. K., & Zhou, Y., Morphic rings and unit regular rings. Journal of Pure and Applied Algebra, 210(2) (2007), 501-510.

[13] Manaviyat, R., Moussavi, A., & Habibi, M., Principally quasi-Baer skew power series rings. Communications in Algebra®, 38(6) (2010), 2164-2176.

[14] Mostafanasab, H., Yetkin, E., Tekir, Ü., & Darani, A. Y., On 2-absorbing primary submodules of modules over commutative rings. Analele Universitatii" Ovidius" Constanta-Seria Matematica, 24(1) (2016), 335-351.

[15] Nagata, M. Local rings. Interscience Tracts in Pure and Appl. Math., 1962.

[16] Payrovi, Sh. and Babaei, S., On 2-absorbing submodules, Algebra Colloq., 19 (2012), 913-920.

[17] Payrovi, S., & Babaei, S., On the 2-absorbing submodules. Iran. J. Math. Sci. Inform, 10(1) (2015), 131-137.

[18] Sevim, E. Ş., Arabaci, T., Tekir, Ü., & Koc, S., On S-prime submodules. Turkish Journal of Mathematics, 43(2) (2019), 1036-1046.

[19] Smith, P. F., Some remark on multiplication modules, Archiv der Mathematik, 50(3) (1988): 223-235.

[20] Smith, P. F., Mappings between module lattices. International Electronic Journal of Algebra, 15(15) (2014), 173-195.

[21] Von Neumann, J., On regular rings. Proceedings of the National Academy of Sciences of the United States of America, 22(12) (1936), 707.

[22] Zhu, H., & Ding, N., Generalized morphic rings and their applications. Communications in Algebra, 35(9) (2007), 2820-2837.

Gülşen Ulucak,
Department of Mathematics,
Gebze Technical University,
Kocaeli, Turkey.
Email: gulsenulucak@gtu.edu.tr

Ünsal Tekir,
Department of Mathematics,
Marmara University,
Istanbul, Turkey.
Email: utekir@marmara.edu.tr

Suat Koç,
Department of Mathematics,
Marmara University,
Istanbul, Turkey.
Email: suat.koc@marmara.edu.tr